

Cheating in Evolutionary Games

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The analysis of models of evolutionary games requires explicit consideration of both evolutionary game rules and mutants which infinitesimally break these rules. For example, the Scotch Auction is an evolutionary game which lacks both a rule-obeying evolutionarily stable strategy and an asymptotically stable polymorphism of rule-obeying strategies. However, an infinitesimal rule-breaking, or cheating, mutant can be found which is an evolutionarily stable strategy against rule-obeying strategies. Such cheating strategies can spread through populations initially playing the Scotch Auction, effectively changing the rules of the game. Moreover, the extent of such rule-change will then tend to increase. Thus, the Scotch Auction is a transient evolutionary game, being the initial point of a seemingly orthogenetic game evolutionary process. This sort of transience suggests that the “progressive” nature of evolution may be due in part to those game features of evolutionary processes which make the success of adaptations relative to the level of extant adaptation among competitors, predators, etc.

1. Introduction

Originally developed by Hamilton (1967) and Maynard Smith & Price (1973) to account for the seemingly anomalous evolution of unequal sex ratios and ritualized conflict, it is becoming apparent that the idea of an evolutionarily stable strategy, or ESS, has wide applicability (cf. Maynard Smith, 1976). The present article seeks to (1) clarify some of the heretofore tacit underpinnings of the ESS concept and (2) proceed to a further extension of evolutionary game theory based on the idea of cheating in evolutionary games.

The fundamental setting for evolutionary game models is a circumscribed, though not necessarily finite, set of genotypes in conflict where the advantages to be gained from individual conflicts, or contests, depend on what the opponent does (Maynard Smith, 1976). From the specified contest situation

defined for each game model, the game theory seeks an ESS which meets the criteria spelled out by Maynard Smith (1974). (These criteria are given later.) The present argument is that *an evolutionary game is defined by specific rules which are not entirely specified by the contest situation*. In part, these rules do arise from the contest pay-offs and penalties. But, in addition, consideration of a specific set of genotypes in an evolutionary game model defines an associated set of allowable alternative game-playing strategies, the complete range of relevant phenotypes. Each of these strategies is defined by a further set of moves and move contingencies, which are thereby "legal", these in turn specifying the remaining rules of the game. So evolutionary game models introduce additional tacit rules when analysis is confined to consideration of a limited set of move possibilities.

As a corollary to this, analysis of an evolutionary game situation where the set of legal moves or move contingencies varies is analysis of different evolutionary games. If this consideration is not borne in mind, confusion as to the status of an ESS will result because ESS derivation is necessarily dependent on model game rules.

Moreover, since game rules depend partly on what the organisms involved do, is it reasonable to assume that they always obey the rules? In Maynard Smith's (1974) War of Attrition, the legal moves were different display durations. Losers stealing disputed food items, players ceasing display contingent on their opponents ceasing, etc would be "cheating". But anti-social behaviour, including such forms of cheating, can in principle be crucial in evolution, as Maynard Smith (1964) has argued. Therefore, it seems necessary to consider the possibility of novel behaviour which, in terms of the previous "state of play", breaks the rules of the game.

Below, the possibilities for cheating in a particular evolutionary game are analysed to illustrate its general importance. The game in question is known as the "Scotch Auction" because it uses some aspects of house-buying in Scotland. There the normal practice is for all interested house-buyers to make a secret offer by a certain deadline. All offers are made known upon the elapse of the deadline, with the highest offer winning the house. Naturally, the other house-buyers do not pay their offer. In the game which we are calling Scotch Auction, on the other hand, all bids are "paid out" in full. Again, only the highest bidder wins the prize.

Below it is shown that the Scotch Auction lacks both a rule-obeying ESS and an asymptotically stable polymorphism of rule-obeying strategies. Let games meeting this condition be defined as pathological. It is shown that natural selection can render some such pathological games non-pathological if infinitesimal cheating is possible. It is also shown that such game-change can produce "progressive" evolution in some game-contexts.

2. A Pathological Game

The Scotch Auction game is effectively a sealed bid auction. Organisms contest a single indivisible pay-off in pairs, each "playing a bid" with a cost directly proportional to the size of the bid. The player making the largest bid wins the pay-off item, paying the full cost of its bid, irrespective of the size of the opponent's bid. The loser pays its full bid too. The single contest pay-offs are as follows. Player *A* plays *m* and player *B* plays *n*, if $m > n$, *A* gets pay-off $V - m$ and *B* gets $-n$, if $m = n$, *A* and *B* both get $\frac{1}{2}V - m$ each and, if $n > m$, *A* gets $-m$ and *B* gets $V - n$.

The remainder of the game rules follow. Confine legal strategies to those which play according to a specific bid probability distribution irrespective of the opponent's bid. (This is the sealed bid rule.) Let the number of contests each player engages in be fixed over all players. Let there be no other avenues by which additional pay-offs may be obtained.

We seek a strategy *I* which is an ESS, satisfying the criteria of Maynard Smith (1974). Any such ESS cannot be invisable by legal strategies. If *I* were simply to play x , $x \in [0, V]$, then any *J* strategy, *J* being a fixed play of $y \in (x, V)$, could invade *I* populations. If $x = V$, *J* with $y = 0$ could invade. Negative x is not defined, and x greater than V also cannot be an ESS. So if there is a suitable *I* conforming to the above rule, it must be a mixed strategy conforming to some probability density function $p(x)$, continuous or otherwise. It is easily seen that any strategy with non-zero probability anywhere on (V, ∞) can be beaten by strategies which are the same except that they have zero probability on (V, ∞) . Therefore bids greater than V will be ignored later.

Let $E(I, J)$ be the pay-off to *I* from contest *I* *v.* *J*. With continuous *I* and *m* any pure strategy, or specific play by a mixed strategist, such that $m \in [0, V]$

$$\begin{aligned} E(m, I) &= \int_0^m (V - m)p(x) dx + \int_m^V (-m)p(x) dx \\ &= V \int_0^m p(x) dx - m. \end{aligned}$$

From this,

$$\frac{d\{E(m, I)\}}{dm} = Vp(m) - 1. \quad (1)$$

Following Maynard Smith (1974), we seek the continuous equilibrium *I* strategy defined by the $p(x)$ for which $E(m, I)$ has the same constant value for all *m*. At this equilibrium, irrespective of *m*,

$$\frac{d\{E(m, I)\}}{dm} = 0,$$

so we obtain

$$p(x) = 1/V, x \in [0, V], \quad (2)$$

and

$$E(m, I) = V \int_0^m (1/V) dx - m = 0. \quad (3)$$

Thus any $E(J, I) = E(I, I) = E(m, I) = 0$, and I defined by equation (2) is a stationary strategy in that any opponent's play does equally poorly.

Is I an ESS? To have an ESS we need (a) $E(I, I) > E(J, I)$ or (b) $E(I, I) = E(J, I)$ and $E(I, J) > E(J, J)$, for all J (Maynard Smith, 1974). Condition (a) is obviously not met, but the first part of condition (b) is met. Is it true that $E(I, J) > E(J, J)$ for all J ? A counter-example is easily produced. In particular,

$$\begin{aligned} E(I, m) &= \int_0^m (-x)(1/V) dx + \int_m^V (V-x)(1/V) dx \\ &= \int_m^V dx - \int_0^V (x/V) dx \\ &= V - m - \frac{1}{2}V \\ &= \frac{1}{2}V - m \\ &= E(m, m) \end{aligned} \quad (4)$$

by definition. Condition (b) is not met, and I cannot be an ESS.

Is an asymptotically stable polymorphism of I and some set of J 's possible? To answer this we must be more specific about the population's genetics. Let the absolute fitness of each individual player be some large constant C plus whatever pay-off is obtained from the game, with the scaling constant for the number of contests relative to contest pay-offs set at unity. Let the population be of infinite size. Let the generations be discrete. Finally, let reproduction be asexual with mutation. Then the frequency of a strategy J in generation $i+1$, say $f_{i+1}(J)$, will be related to $f_i(J)$ according to the usual equation,

$$f_{i+1}(J) = \frac{W(J)}{\bar{W}} f_i(J). \quad (5)$$

where $W(J)$ is the absolute fitness of J strategists and \bar{W} is the mean fitness. Let $\Delta f(J)$ be defined as $f_{i+1}(J) - f_i(J)$.

Consider a mutant pure strategist, $m \in [0, V]$, in an otherwise monomorphic population of I 's. In this case, using equations (3) and (4),

$$\begin{aligned} W(I) &= C + [1 - f_i(m)]E(I, I) + f_i(m)E(I, m) \\ &= C + f_i(m)(\frac{1}{2}V - m) \end{aligned}$$

and

$$\begin{aligned} W(m) &= C + [1 - f_i(m)]E(m, I) + f_i(m)E(m, m) \\ &= C + f_i(m)(\frac{1}{2}V - m) \\ &= W(I). \end{aligned}$$

Independent of $f_i(m)$, $\bar{W} = W(I) = W(m)$ so $\Delta f(I) = \Delta f(m) = 0$. This is a neutrally stable polymorphism, allowing $f_i(m)$ to fluctuate over the interval $(0, 1)$. Thus an I and m polymorphism cannot be asymptotically stable.

Since m can remain in the population indefinitely, what would be the fate of a second pure strategy? Now the population consists of I , m , and n , m and $n \in (0, V)$. Label m and n such that $m > n$. The absolute fitnesses, again using equations (3) and (4), are as follows.

$$\begin{aligned} W(I) &= C + \{1 - [f_i(m) + f_i(n)]\}E(I, I) + f_i(m)E(I, m) + f_i(n)E(I, n) \\ &= C + f_i(m)(\frac{1}{2}V - m) + f_i(n)(\frac{1}{2}V - n). \\ W(m) &= C + f_i(m)(\frac{1}{2}V - m) + f_i(n)(V - m) \\ &= W(I) + f_i(n)(\frac{1}{2}V + n - m). \\ W(n) &= C + f_i(m)(-n) + f_i(n)(\frac{1}{2}V - n) \\ &= W(I) + f_i(m)(m - \frac{1}{2}V - n). \end{aligned}$$

The mean fitness \bar{W} simplifies to $W(I)$, independent of $f_i(m)$ and $f_i(n)$. From equation (5), we find

$$\begin{aligned} \Delta f(I) &= 0, \\ \Delta f(m) &= \frac{f_i(n)(\frac{1}{2}V + n - m)}{W(I)}, \end{aligned}$$

and

$$\Delta f(n) = \frac{f_i(m)(m - \frac{1}{2}V - n)}{W(I)}.$$

At equilibrium, $f_i(n) = 0$, $f_i(m) = 0$, or $m = \frac{1}{2}V + n$. Equilibrium with m and n non-zero requires that $m = \frac{1}{2}V + n$. Again, this is a neutrally stable polymorphism, with no constraints on the frequencies of I , m , or n , the result holding with $f_i(I) = 0$. If m is less than $\frac{1}{2}V + n$, $f_i(m)$ increases with i . If m is greater than $\frac{1}{2}V + n$, $f_i(m)$ decreases with i . In either case, $f_i(n)$ changes in exactly the same proportions in the opposite direction.

Consider the invasion of a third pure strategy in a population which is at the neutrally stable equilibrium found above. Now we have a population which consists of I , l , m and n , labelled so that $l > m > n$, with one of the

(x, y) pairs, (l, m) , (l, n) , and (m, n) , obeying the relation $x = \frac{1}{2}V + y$. Again, we find that $\bar{W} = W(I)$. The dynamics are given by

$$\begin{aligned}\Delta f(I) &= 0, \\ \Delta f(l) &= \frac{f_i(m)(\frac{1}{2}V + m - l) + f_i(n)(\frac{1}{2}V + n - l)}{W(I)}, \\ \Delta f(m) &= \frac{f_i(l)(l - \frac{1}{2}V - m) + f_i(n)(\frac{1}{2}V + n - m)}{W(I)}\end{aligned}$$

and

$$\Delta f(n) = \frac{f_i(l)(l - \frac{1}{2}V - n) + f_i(m)(m - \frac{1}{2}V - n)}{W(I)}.$$

If $l = \frac{1}{2}V + m$, four-strategy equilibrium requires that $n = l - \frac{1}{2}V = m - \frac{1}{2}V$, or $l = m$. Likewise, if $l = \frac{1}{2}V + n$, equilibrium requires $m = l$, and if $m = \frac{1}{2}V + n$, equilibrium requires $m = n$. Thus, for distinct l , m , and n , there is no four-strategy equilibrium involving three pure strategies, so there is certainly no asymptotically stable polymorphism involving such strategies.

So far it has been shown that populations with I and up to two pure strategies can be neutrally stable, but that in populations with three pure strategies, two of which can co-exist, there is no equilibrium, whether or not $f_i(I)$ is strictly positive. Since I is legal, it ensures at most neutral stability in these particular pure strategy cases. What occurs in the general case? In Appendices A and B, it is shown that (1) any number of discrete or continuous probability strategies can co-exist in equilibrium, (2) all equilibria are at most neutrally stable if not unstable, and (3) there is no ESS against I . Thus there is no ESS *and* no asymptotically stable polymorphism among discrete or continuous probability rule-obeying strategies; so the Scotch Auction is a pathological game.

3. A Cheating ESS

A mutant which infinitesimally breaks the rules of the game can be shown to be an ESS against all rule-obeying strategies. Let I^* be a mutant I strategist which normally plays the usual I moves, but in a proportion r of its encounters with non- I^* strategists will play $x + (V - x)q$, $q \in (0, \frac{1}{2})$, against an opponent's bid of x . Obviously, I^* must have information concerning its opponent's bid, but only r of the time. Making r close to zero allows for a reasonable infinitesimal rule-change.

First, let us return to the m and I polymorphic equilibrium case. We find

$$\begin{aligned} E(I, I^*) &= r \int_0^V (-x/V) dx \\ &= \frac{-rV}{2} < 0 = E(I, I). \end{aligned}$$

I is not an ESS against I^* , nor is it neutral, because

$$\begin{aligned} E(I^*, I) &= r \int_0^V (1/V)[V-x-(V-x)q] dx \\ &= \frac{r(1-q)}{V} \int_0^V V-x dx \\ &= \frac{(1-q)rV}{2} \end{aligned} \quad (6)$$

and

$$E(I^*, I^*) = E(I, I) = 0 < E(I^*, I). \quad (7)$$

In addition,

$$E(m, I^*) = -mr$$

and

$$\begin{aligned} E(I^*, m) &= (1-r)(\frac{1}{2}V-m) + r[V-m-(V-m)q] \\ &= (\frac{1}{2}V-m) + r[\frac{1}{2}V-(V-m)q] > E(m, m). \end{aligned} \quad (8)$$

Using equations (6), (7), and (8), we see that I^* will increase in frequency and that it is an ESS in an I^* , I , and m contest. The extension to a multiplicity of pure strategies is clear, and I^* will still be an ESS.

Is there any mixed strategy which allows $E(J, I^*) \geq E(I^*, I^*)$? In the discrete probability case,

$$\begin{aligned} E(J, I^*) &= (1-r)E(J, I) + r \sum_i p_i(-x_i) \\ &= -r\bar{x}_J < 0 = E(I^*, I^*). \end{aligned}$$

In the continuous probability case,

$$\begin{aligned} E(J, I^*) &= (1-r)E(J, I) + r \int_0^V p(x)(-x) dx \\ &= -r\bar{x}_J < 0 = E(I^*, I^*). \end{aligned}$$

So I^* is an ESS against any set of rule-obeying strategies.

For I_1^* and I_2^* differing only in that $r_1 > r_2$, it is apparent that $W(I_1^*) > W(I_2^*)$ so long as there are some strategists other than I_i^* in the population.

In the long run, assuming intermittent occurrence of such strategies—perhaps due to immigration from other populations, the mean value of r will tend to increase. Likewise, selection under these conditions will act to reduce the magnitude of q . Hence any particular I_i^* will not be an ESS against other cheating strategists, and the game rules will continue to change until r is at unity and q is arbitrarily close to zero.

4. Discussion

We have seen that (1) the Scotch Auction is pathological, (2) cheating mutants can take over populations of Scotch Auction rule-obeying strategists, and (3) such cheating strategists will become more efficient, wholly undermining the Scotch Auction's sealed-bid rule. Three issues remain. Firstly, what sort of biological situation would conform to the Scotch Auction? Secondly, what will happen after the sealed-bid has been undermined? And thirdly, what does the Scotch Auction case show about cheating in evolutionary games generally?

One case in which the Scotch Auction might arise is growth competition among plants. Here the extra pay-off would be canopy exposure to the sun, with shaded individuals suffering an appreciable, but not disastrous, fitness loss. The bid would be the amount of energy invested in growth. A tie would occur when both reach the canopy, each somewhat crowding the other's top leaves. The cheating strategist would then be one which waits until its opponent has finished growth, and then grows somewhat more. Presumably some photosensitive growth-determination mechanism could produce this move contingency. But how do two such cheaters deal with each other? Perhaps by waiting until rule-obeying strategists would have completed growth, and going ahead as rule-obeying strategists if left unshaded. So the above analysis could be applicable to this case.

Turning to the second issue, in this example further cheating, with one cheater occasionally waiting until the other cheaters have finished growth and then just beating them, could evolve, depending on constraints to further growth delay. Alternatively, between cheating strategists, further cheating could entail lopsided growth away from the opponent, mitigating the importance of contest loss by reducing the opponent's shading of the cheater. The success of this rule change would depend on the costs of the additional horizontal growth relative to the shading reduction benefit. Evidently, the possibilities for game "tinkering" (Jacob, 1977) will depend on the total context of evolutionary constraints. Unfortunately, the richness of such contexts among real-world populations precludes mathematical analysis. Thus nothing can be said with generality about the further evolution of cheating.

However, it is clear that the Scotch Auction sealed-bid rule will be a *transient* feature of the game so long as there are just infinitesimal opportunities for cheating in populations initially obeying this rule.

Now consider a contrasting case, that of the Hawk–Dove conflict from Maynard Smith & Price (1973). Treat escalated conflict behaviour as cheating in a nominal Dove–Dove conventionalized contest. Initially the Hawk cheater will spread through the population, even if it cheats only infinitesimally to begin with. However, if the benefits from each contest pay-off are less than the expected losses due to injury in escalated combat, Retaliator can change the rules again, spread through the population, eliminate Hawk, and re-establish the conventionalized contest (Maynard Smith & Price, 1973). Thus game rule changes may act to counter cheating which violates specific game rules. Though Retaliator does change the Dove–Dove game by introducing a “game policing” move contingency, the conventionalized feature of the contest situation is conserved. Thus the rule-change, or “game evolution”, process may act to sustain some game features.

Clearly, it would be difficult to assess under what conditions a particular game feature will persist in the face of cheating. It may be easier to identify transient game features than to pin-point those which can be sustained. Indeed, transience may be the long-term rule, rather than the exception.

In terms of long-term evolution, perhaps some cases of “sustained progress” may be due to this transience. The semblance of orthogenesis might arise from consistent game evolution due to evolving game logics which foster cheating mutations to particular game features. Some phylogenetic consistencies in macroevolutionary trends might thereby be due to game logic consistencies which are shared within the phylogenetic lines. Such “orthogenetic progress” would be exhibited within taxa playing the Scotch Auction game, due to the progressive increase in the efficiency of cheating. To give a well-known example, the fairly steady evolution of the horse as a large cursorial animal may have been due to a succession of transient interspecific evolutionary games with a cursorial predator where each was always getting just a bit faster than the other, the lack of a sustainable equilibrium perpetuating the process. In part, one might attribute the relative rarity of “living fossil” species to the transient nature of evolutionary games in the face of intermittent opportunities for successful cheating. If cheating unavoidably tends to drive game evolution on, some sort of evolutionary “progress” would only rarely die out before extinction.

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APPENDIX A

Dynamics with Discrete Probability Strategies

Let I be as in the text. Mixed strategies J_j are each defined by pure strategies x_{ji} ($i = 1, 2, \dots, n_j$) and associated probabilities of play p_{ji} , with

$$\sum_i p_{ji} = 1.$$

For convenience, let $x_{ji} < x_{j,i+1}$, for all j and all i .

First we find that

$$E(I, J_j) = E(J_j, J_j). \quad (\text{A1})$$

Straightforwardly,

$$E(I, J_j) = \sum_i p_i (\frac{1}{2}V - x_i) = \frac{1}{2}V - \bar{x}_j.$$

Separating out the expected pay-offs from the play of each x_{ji} ,

$$\begin{aligned} E(J_j, J_j) &= \sum_{i=1}^{n_j} p_{ji}^2 (\frac{1}{2}V - x_{ji}) + \sum_{i=1}^{n_j} \left(\sum_{k=i+1}^{n_j} p_{ji} p_{jk} (-x_{ji}) \right) + \\ &\quad + \sum_{i=1}^{n_j} \left(\sum_{k=1}^{i-1} p_{ji} p_{jk} (V - x_{ji}) \right) \\ &= \frac{1}{2}V \left\{ \sum_{i=1}^{n_j} p_{ji}^2 + 2 \sum_{i=1}^{n_j} p_{ji} \left(\sum_{k=1}^{i-1} p_{jk} \right) \right\} - \\ &\quad - \sum_{i=1}^{n_j} p_{ji}^2 x_{ji} + \sum_{i=1}^{n_j} p_{ji}^2 x_{ji} - \sum_{i=1}^{n_j} p_{ji} x_{ji} \\ &= \frac{1}{2}V (\sum_i p_{ji}) (\sum_i p_{ji}) - \bar{x}_{J_j} = \frac{1}{2}V - \bar{x}_{J_j}. \end{aligned}$$

Next we need

$$\begin{aligned} E(J_k, J_l) + E(J_l, J_k) - E(J_k, J_k) - E(J_l, J_l) &= 0. \quad (\text{A2}) \\ E(J_k, J_l) + E(J_l, J_k) - E(J_k, J_k) - E(J_l, J_l) \\ &= \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} p_{ki} p_{lj} [E(x_{ki}, x_{lj}) + E(x_{lj}, x_{ki})] + \bar{x}_{J_k} + \bar{x}_{J_l} - V \\ &= \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} p_{ki} p_{lj} (V - x_{ki} - x_{lj}) + \bar{x}_{J_k} + \bar{x}_{J_l} - V \\ &= V \left(\sum_{i=1}^{n_k} p_{ki} \right) \left(\sum_{j=1}^{n_l} p_{lj} \right) - V + \bar{x}_{J_k} - \bar{x}_{J_k} + \bar{x}_{J_l} - \bar{x}_{J_l}. \end{aligned}$$

Now we find a sufficient, but not necessary, condition for neutral stability, with $n J_j$ strategies. For all j :

$$E(J_i, J_j) = E(J_j, J_j) \quad i = 1, \dots, n. \quad (\text{A3})$$

This has been found for $n = 2$ and 3 , and it is true trivially for $n = 1$. Assume that there is some maximum finite value of n for which condition (A3) applies. We show that (A3) then applies for $n+1$ as well, so (A3) applies for any n by induction.

In the i th generation, with $n+1 J_j$ strategists the absolute fitnesses are as follows, assuming (A3) for $j = 1, \dots, n$.

$$W(I) = C + \sum_{k=1}^n f_i(J_k)E(J_k, J_k) + f_i(J_{n+1})E(J_{n+1}, J_{n+1}).$$

For $j = 1, \dots, n$

$$W(J_j) = C + \sum_{k=1}^n f_i(J_k)E(J_k, J_k) + f_i(J_{n+1})E(J_j, J_{n+1}).$$

$$W(J_{n+1}) = C + \sum_{k=1}^n f_i(J_k)E(J_{n+1}, J_k) + f_i(J_{n+1})E(J_{n+1}, J_{n+1})$$

The mean fitness is as follows.

$$\begin{aligned} \bar{W} &= \left[1 - \sum_{k=1}^{n+1} f_i(J_k) \right] W(I) + \\ &+ \sum_{k=1}^n f_i(J_k) \{ W(I) + f_i(J_{n+1}) [E(J_k, J_{n+1}) - E(J_{n+1}, J_{n+1})] \} \\ &+ f_i(J_{n+1}) \left\{ W(I) + \sum_{k=1}^n f_i(J_k) [E(J_{n+1}, J_k) - E(J_k, J_k)] \right\} \\ &= W(I) + f_i(J_{n+1}) \sum_{k=1}^n f_i(J_k) [E(J_k, J_{n+1}) + E(J_{n+1}, J_k) - E(J_k, J_k) - \\ &\quad - E(J_{n+1}, J_{n+1})]. \\ &= W(I). \end{aligned}$$

From this, it follows that

$$\begin{aligned} \Delta f(I) &= 0, \\ \Delta f(J_j) &= \frac{f_i(J_{n+1}) [E(J_j, J_{n+1}) - E(J_{n+1}, J_{n+1})]}{W(I)} \end{aligned}$$

for $j = 1, \dots, n$ and

$$\Delta f(J_{n+1}) = \frac{\sum_{k=1}^n f_i(J_k) [E(J_{n+1}, J_k) - E(J_k, J_k)]}{W(I)}.$$

With $f_i(J_{n+1})$ strictly positive, the $\Delta f(J_j)$ vanish if and only if $E(J_j, J_{n+1}) = E(J_{n+1}, J_{n+1})$ for $j = 1, \dots, n$. Using equation (A2), this also gives $E(J_{n+1}, J_k) = E(J_k, J_k)$. Note that this result does not require positive $f_i(I)$.

Strategies satisfying sufficiency condition (A3) can always be found. For example, strategies defined by probability distributions which are probabilistic linear combinations of the extant J_j and I will satisfy (A3) because each of their constituent strategic options do so. Therefore any number of strategies can co-exist in neutral equilibrium.

When condition (A3) is *not* met, with I and any n legal J_j strategies satisfying (A1) and (A2).

$$\begin{aligned} W(J_j) &= C + \sum_{k=1}^n f_i(J_k)E(J_j, J_k) \\ &= W(I) + \sum_{k=1}^n f_i(J_k)[E(J_j, J_k) - E(J_k, J_k)]. \end{aligned}$$

It follows that

$$\begin{aligned} \bar{W} &= W(I) + \sum_{j=1}^n f_i(J_j) \left\{ \sum_{k=1}^n f_i(J_k) [E(J_j, J_k) - E(J_k, J_k)] \right\} \\ &= W(I) + \sum_{j=1}^n f_i(J_j) \left\{ \sum_{k=1}^{j=1} f_i(J) [E(J_j, J_k) + E(J_k, J_j) - E(J_k, J_k) - \right. \\ &\quad \left. - E(J_j, J_j)] \right\} \\ &= W(I). \end{aligned}$$

Thus it is always true, when (A1) and (A2) hold, that $\Delta f(I)$ is zero, so equilibria are at most neutrally stable since I is legal. Therefore I can always increase in frequency by mutation alone, and no other legal strategy can be an ESS.

APPENDIX B

Dynamics with Continuous Probability Strategies

Consider mixed strategies J_j defined by $p_j(x)$ giving the probability that strategist J_j plays bid x , where $p_j(x)$ is always non-negative and

$$\int_{-\infty}^{\infty} p_j(x) dx = \int_0^V p_j(x) dx = 1.$$

If results (A1) and (A2) hold for continuous probability J_j , (A3) necessarily follows because its proof applies to any set of J_j satisfying (A1) and (A2).

First we find that (A1) remains true. Once more,

$$\begin{aligned} E(I, J_j) &= \int_0^V p_j(x) E(I, x) dx \\ &= \int_0^V p_j(x) (\frac{1}{2}V - x) dx \\ &= \frac{1}{2}V - \bar{x}_{J_j}. \end{aligned}$$

And

$$\begin{aligned} E(J_j, J_j) &= \int_0^V p_j(x) \left[\int_0^x p_j(u) (V - x) du + \int_x^V p_j(u) (-x) du \right] dx \\ &= V \int_0^V p_j(x) \int_0^x p_j(u) du dx - \bar{x}_{J_j}, \\ &= V \int_0^V \int_0^V p_k(x) p_j(u) \chi_{(u < x)} du dx - \bar{x}_{J_j}, \end{aligned}$$

where $\chi_{(-)}$ is a Boolean function taking value 1 if the statement in parentheses is true and having value 0 otherwise. Relabelling,

$$\begin{aligned} &\int_0^V \int_0^V p_j(x) p_j(u) \chi_{(u < x)} du dx \\ &= \int_0^V \int_0^V p_j(u) p_j(x) \chi_{(x < u)} dx du \\ &= \int_0^V p_j(x) \int_0^V p_j(u) \chi_{(x < u)} du dx \\ &= \int_0^V p_j(x) \int_x^V p_j(u) du dx. \end{aligned}$$

In addition,

$$\begin{aligned} 1 &= \int_0^V \int_0^x p_j(x) p_j(u) du dx + \int_0^V \int_x^V p_j(x) p_j(u) du dx \\ &= 2 \int_0^V \int_0^x p_j(x) p_j(u) du dx. \end{aligned}$$

$$\text{So } E(J_j, J_j) = \frac{1}{2}V - \bar{x}_{J_j}.$$

Result (A2) follows directly,

$$\begin{aligned}
 & E(J_k, J_l) + E(J_l, J_k) - E(J_l, J_l) - E(J_k, J_k) \\
 = & \int_0^V \int_0^V p_k(x)p_l(y)(V-x-y) \, dx \, dy + \bar{x}_{J_k} + \bar{x}_{J_l} - V \\
 = & V \int_0^V \int_0^V p_k(x)p_l(y) \, dx \, dy - V \\
 & + \bar{x}_{J_k} - \int_0^V p_k(x)x \left(\int_0^V p_l(y) \, dy \right) \, dx \\
 & + \bar{x}_{J_l} - \int_0^V p_l(y)y \left(\int_0^V p_k(x) \, dx \right) \, dy \\
 = & 0.
 \end{aligned}$$